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INVARIANT INTEGRALS FOR THERMO-VISCOELASTICITY AND APPLICATIONS

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Abstract—Invariant integrals are derived for coupled time dependent thermoviscoelastic solids with spatially varying moduli. These integrals are used to obtain crack tip stress and displacement fields for certain special strip problems where constant temperatures and displacements are applied to the strip sides. The steady thermoelastic equations can be viewed as a limit of the time dependent coupled equations and hence the above formulations can be used to obtain results. However, no direct formulation with the desired properties seems possible for the steady equations. Hence, to obtain results for steady thermoelastic problems we derive a variational principle and invariant integrals for a "pseudo" set of thermoelastic equations which can be shown to reduce to the well known equations in an appropriate limit. Applications of these new integrals are discussed.

1. INTRODUCTION

Recently, Wilson and Yu[1] and Gurtin[2] have considered path independent integrals for static linear isotropic thermoelasticity. In [1] the integral is not strictly path independent since the expressions used involve a nonvanishing volume integral (surface integral for two dimensional applications). The integral in [2] suffers from the disadvantage that the integrand is not zero along the crack faces. It is stressed in Atkinson[3] that this last point is worth checking since when the integrand is not zero on the crack faces, the end points of a contour enclosing the crack tip cannot be slid along the crack without changing the value of the integral.

To overcome the difficulties present in the above formulations we derive here alternative invariant integrals. Moreover our integrals are valid for *coupled time dependent thermo viscoelastic solids*, and in addition in favorable situations the integrands will be zero on the crack face. It turns out that these integrals, while suitably invariant for the coupled theory, are not invariant for the uncoupled theory. Nevertheless, it should be possible to perform certain calculations using the full coupled invariant integral and then obtain results for time independent, uncoupled theories by a limiting process. Some consideration of this is given later. To obtain these invariant integrals for the coupled time dependent thermoviscoelastic theory we first Laplace transform the coupled equations of motion and then construct a Lagrangian which produces the transformed equations as its Euler–Lagrange equations. Once this Lagrangian (L) is constructed it is straightforward to construct invariant integrals in terms of a tensor

$$P_{lj} = \frac{\partial L}{\partial \bar{u}_{i,j}} \bar{u}_{i,l} + \frac{\partial L}{\partial \bar{\theta}_{,j}} \bar{\theta}_{,l} - L\delta_{lj} \tag{1.1}$$

where $\bar{u}_i$ are the Laplace transformed displacement components and $\bar{\theta}$ is the Laplace transformed temperature. To do this we follow a prescription outlined by Eshelby[4]. It should be noted that in [4] the argument is phrased in terms of the Energy-momentum tensor whereas no such physical description is applicable to the tensor P_{lj} defined in eqn (1.1) since this is defined in the Laplace transformed domain.

The tensor P_{lj} defined in (1.1) can be directly useful as a stress-analysis tool in certain special situations. To illustrate this we consider in Section 3 the problem of a semi-infinite crack in an infinite strip with spatially independent boundary conditions on the strip sides. For such boundary conditions it is possible by means of a path independent integral to determine the singular field of the crack tip from calculations made as x_1 tends to plus or minus infinity in the

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strip. Calculations of this kind have been made by Nilsson[5] for a homogeneous viscoelastic strip and by Atkinson[6] for inhomogeneous strips, i.e. strips in which the moduli may vary spatially in a direction perpendicular to the crack direction. Each of these applications is to situations where the crack is stationary in a time dependent stress field. Situations involving steadily moving cracks in displacement loaded inhomogeneous strips have been considered by Atkinson[7] and this and other applications are discussed in [8]. Since in these applications the moduli vary with position in a direction perpendicular to the crack direction the calculations of [6, 7] include the case of a crack lying at the interface between two dissimilar media. A discussion of such an interface crack in the stationary bimaterial elastic case has been discussed independently by Smelser and Gurtin[9] although explicit calculations were not considered.

The calculations of Section 3 are set up to deal with the general coupled case with a variety of spatially independent boundary conditions on the strip sides. However, in certain situations it is not possible to extract explicit information about stress intensity factors even in principle. For example if the temperature boundary conditions are such that singular thermal gradients are anticipated at the crack tip then the integral considered in Section 3 will only give a linear combination of the squares of a stress intensity factor and the thermal intensity factor (coefficient of thermal gradient times $r^{1/2}$ ahead of the crack tip). This is reminiscent of what happens in the bimaterial elastic case. For other boundary conditions however, explicit determination of intensity factors should be possible even in the coupled case. The algebra involved is quite complicated and so various simplified cases are considered in Section 3. The results are given in terms of the Laplace transform variable so further work is required to give real time results. Certain limiting large time results can be obtained relatively easily and are discussed in Section 5.

We have noted at the beginning of this introduction that invariant integral formulations for static linear thermoelasticity are either not suitably invariant or are not zero on the crack faces, and hence cannot be used for the applications envisaged here. The full coupled theory outlined in Section 2 and 3 should however be applicable to static situations as the limit $t \to \infty$ (or $p \to 0$) of a transient situation. Although this is correct it is perhaps more complicated than neccessary if only a static analysis is required. As a simple alternative we derive in Section 4 a variational principle for a set of "pseudo" static thermoelastic equations which reduce to the usual ones when a certain coefficient tends to infinity. Calculations of the kind discussed in Section 3 can then be made directly in real time for steady thermoelastic situations.

Finally in Section 5 some generalizations are discussed and a comparison made of the results of Sections 3 and 4. It should be noted that the invariant integral approach used here leads to a determination of the square of the intensity factors (in favorable cases). The sign of the intensity factor must be determined by other considerations, e.g. symmetry. Furthermore since the stress intensity factor is a local quantity at the crack tip, this only determines the opening of the crack near the tip. To determine in critical cases the opening of the full extent of the crack a full analysis is required.

2. INVARIANT INTEGRALS FOR COUPLED, TIME DEPENDENT, THERMOELASTICITY AND THERMOVISCOELASTICITY

For brevity we begin with the equations of the linear theory of coupled thermoviscoelasticity as given for example in Christensen[10]. The corresponding equations for the coupled thermoelastic case can be easily seen as a limiting case. Without body forces the relevant equations can be written

$$\sigma_{ij,j} = \rho \partial^2 u_i / \partial t^2 \tag{2.1}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.2}$$

(the commas denoting partial differentiation)

and

$$\sigma_{ij} = \int_0^t G_{ijkl}(t-\tau)\frac{\partial \epsilon_{kl}(\tau)}{\partial \tau}\,\mathrm{d}\tau - \int_0^t \phi_{ij}(t-\tau)\frac{\partial \theta(\tau)}{\partial \tau}\,\mathrm{d}\tau \quad \text{(anisotropic)} \tag{2.3}$$

with

$$s_{ij} = \int_0^t G_1(t-\tau)\frac{\partial e_{ij}(\tau)}{\partial \tau}\,d\tau \quad \text{(isotropic)} \tag{2.4}$$

$$\sigma_{kk} = \int_0^t G_2(t-\tau)\frac{\partial \epsilon_{kk}(\tau)}{\partial \tau}\,d\tau - 3\int_0^t \phi(t-\tau)\frac{\partial \theta(\tau)}{\partial \tau}\,d\tau.$$

The deviatoric components s_{ij} and e_{ij} are defined as

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \qquad s_{ii} = 0$$

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\delta_{ij}\epsilon_{kk}, \qquad e_{ii} = 0. \tag{2.5}$$

The summation convention is applied to repeated indices and i, j and k take the values 1, 2 or 3.

The heat conduction equation is given by

$$\frac{k_{ij}}{T_0}\theta_{,ij} = \frac{\partial}{\partial t}\int_0^t m(t-\tau)\frac{\partial \theta(\tau)}{\partial \tau}\,d\tau + \frac{\partial}{\partial t}\int_0^t \phi_{ij}(t-\tau)\frac{\partial \epsilon_{ij}(\tau)}{\partial \tau}\,d\tau \quad \text{(anisotropic)} \tag{2.6}$$

or

$$\frac{k}{T_0}\theta_{,ii} = \frac{\partial}{\partial t}\int_0^t m(t-\tau)\frac{\partial \theta(\tau)}{\partial \tau}\,d\tau + \frac{\partial}{\partial t}\int_0^t \phi(t-\tau)\frac{\partial \epsilon_{kk}(\tau)}{\partial \tau}\,d\tau \quad \text{(isotropic).} \tag{2.7}$$

In the above equations k_{ij}, or k, $m(t)$ and $\phi_{ij}(t)$ or $\phi(t)$ are mechanical properties of the material. Similarly the functions $G_i(t)$, $G_{ijkl}(t)$ are the relaxation moduli. The reader is referred to [10] for more details of this formulation. $\theta(\tau)$ denotes the infinitesimal temperature deviation from the base temperature T_0.

Suppose now initial conditions are considered in which

$$\theta(t) = u_i(t) = \sigma_{ij}(t) = 0 \qquad \text{for} \quad t < 0. \tag{2.8}$$

Then Laplace transforming eqns (2.1)–(2.8) gives the equations

$$\bar{\sigma}_{ij,j} = \rho p^2 \bar{u}_i \tag{2.9}$$

$$\bar{\epsilon}_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}) \tag{2.10}$$

$$\bar{\sigma}_{ij} = p\bar{G}_{ijkl}\bar{\epsilon}_{kl} - p\bar{\phi}_{ij}\bar{\theta} \quad \text{(anisotropic)} \tag{2.11}$$

or

$$\bar{s}_{ij} = p\bar{G}_1\bar{e}_{ij}, \quad \bar{\sigma}_{kk} = p\bar{G}_2\bar{\epsilon}_{kk} - 3p\bar{\phi}\bar{\theta} \quad \text{(isotropic)}$$

$$\bar{s}_{ij} = \bar{\sigma}_{ij} - \frac{1}{3}\delta_{ij}\bar{\sigma}_{kk}, \quad \bar{e}_{ij} = \bar{\epsilon}_{ij} - \frac{1}{3}\delta_{ij}\bar{\epsilon}_{kk} \tag{2.12}$$

and for the temperature

$$(k_{ij}/T_0)\bar{\theta}_{,ij} = p^2\bar{m}\bar{\theta} + p^2\bar{\phi}_{ij}\bar{\epsilon}_{ij} \quad \text{(anisotropic)} \tag{2.13}$$

or

$$(k/T_0)\bar{\theta}_{,ii} = p^2\bar{m}\bar{\theta} + p^2\bar{\phi}\bar{\epsilon}_{kk} \quad \text{(isotropic).} \tag{2.14}$$

The notation

$$\bar{f} = \int_0^\infty e^{-pt} f(t)\,dt \tag{2.15}$$

has been used for the Laplace transform. Note that if the viscoelastic moduli $p\bar{G}_{ijkl}$, etc. are replaced by constants independent of p and similarily $p\bar{\phi}_{ij}$ are replaced by material constants independent of p then the above equations reduce to the time transformed thermoelastic equations.

It is convenient for our subsequent formulation to define a tensor $\bar{t}_{ij}$ as

$$\bar{t}_{ij} = p\bar{G}_{ijkl}\bar{\epsilon}_{kl} \quad \text{(anisotropic)} \tag{2.16}$$

or

$$\bar{t}_{ij} = p\bar{G}_1\bar{e}_{ij} + \frac{1}{3}\delta_{ij}p\bar{G}_2\bar{\epsilon}_{kk} \quad \text{(isotropic).} \tag{2.17}$$

Then (2.11) and (2.12) can be written

$$\begin{aligned} \bar{\sigma}_{ij} &= \bar{t}_{ij} - p\bar{\phi}_{ij}\bar{\theta} \quad \text{(anisotropic)} \\ \bar{\sigma}_{ij} &= \bar{t}_{ij} - p\bar{\phi}\bar{\theta}\delta_{ij} \quad \text{(isotropic).} \end{aligned} \tag{2.18}$$

Our procedure for finding invariant integrals will be to first find a Lagrangian for the above system of equations and then to apply systematically a procedure outlined in [4] for generating invariant integrals via the energy momentum tensor. The arguments outlined in [4] have in mind applications to situations where the invariant integrals have physical significance such as the force on a defect. However, it is possible to apply the procedure even when the resulting integrals may not themselves have any physical significance but may nevertheless serve as (weakly) useful tools in stress analysis.

One possible Lagrangian for the anisotropic eqns (2.9)–(2.11) and (2.13) can be written

$$L = -\frac{1}{2}\bar{t}_{ij}\bar{\epsilon}_{ij} - \rho\frac{p^2}{2}\bar{u}_i\bar{u}_i + p\bar{\phi}_{ij}\bar{\theta}\bar{u}_{i,j} + \frac{1}{2T_0 p}k_{ij}\bar{\theta}_{,i}\bar{\theta}_{,j} + \frac{1}{2}p\bar{m}\bar{\theta}^2 \tag{2.20}$$

where $\bar{t}_{ij}$ is defined in (2.16).

To check this recall that if (2.9)–(2.11) and (2.13) are to be generated by a variational principal of the form, $\int L\,dV$ stationary, then they must result from the Euler–Lagrange equations:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \bar{u}_{i,j}}\right) - \frac{\partial L}{\partial \bar{u}_i} = 0 \qquad i = 1, 2, 3 \tag{2.21}$$

(the summation convention applying to the index j)

and

$$\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \bar{\theta}_{,j}}\right) - \frac{\partial L}{\partial \bar{\theta}} = 0. \tag{2.22}$$

Note that in the above derivation the dependence $L(\bar{u}_{i,j}, \bar{u}_i, \bar{\theta}_{,j}, \bar{\theta}, x_j)$ is considered and the symmetries $\bar{t}_{ij} = \bar{t}_{ji}$, $\bar{\phi}_{ij} = \bar{\phi}_{ji}$, $k_{ij} = k_{ji}$ are assumed. Furthermore, eqns (2.9)–(2.11) and (2.13) will still follow from (2.20), (2.21) and (2.22) even if the coefficients $\bar{G}_{ijkl}$, $\bar{\phi}_{ij}$, $\bar{m}$ and k_{ij} depend explicitly on the x_j (i.e. spatially inhomogeneous media). In this last case the l.h.s. of eqn (2.6) and subsequent equations would be replaced by $(k_{ij}\theta_{,i})_{,j}$.

If we now define the tensor

$$P_{lj} = \frac{\partial L}{\partial \bar{u}_{i,j}} \bar{u}_{i,l} + \frac{\partial L}{\partial \bar{\theta}_{,j}} \bar{\theta}_{,l} - L\delta_{lj} \tag{2.23}$$

then it can be shown in an elementary manner (see [4, 6]) that

$$P_{lj,j} = -\left(\frac{\partial L}{\partial x_l}\right)_{\text{explicit}}. \tag{2.24}$$

The subscript explicit indicates that the partial differentiation is with respect to the terms in x_l which appear explicitly in the Lagrangian. From (2.23) and (2.20) we deduce that

$$P_{lj} = -(\bar{t}_{ij} - p\bar{\theta}\bar{\phi}_{ij})\bar{u}_{i,l} + \frac{k_{ij}}{T_0 p} \bar{\theta}_{,i}\bar{\theta}_{,l} - L\delta_{lj}$$

hence

$$P_{lj} = -\bar{\sigma}_{ij}\bar{u}_{i,l} + \frac{k_{ij}}{T_0 p} \bar{\theta}_{,i}\bar{\theta}_{,l} - L\delta_{lj}. \tag{2.25}$$

Given P_{lj} it is possible to derive invariant integrals as, for example,

$$F_l = \int_S P_{lj} \, \mathrm{d}s_j \tag{2.26}$$

where $\mathrm{d}s_j = n_j \, \mathrm{d}s$, n_j being the outward normal to the surface element $\mathrm{d}S$. Taking S as a closed surface in the body which does not enclose any singularities a straightforward application of the divergence theorem leads to

$$F_l = \int_V P_{lj,j} \, \mathrm{d}V$$

and this equals zero from (2.24) provided L does not depend explicitly on x_l.

For the isotropic problem designated by eqns (2.9), (2.10), (2.12) and (2.14) we can write

$$L = -\frac{1}{2} \bar{t}_{ij}\bar{\epsilon}_{ij} - \frac{\rho p^2}{2} \bar{u}_i\bar{u}_i + p\bar{\phi}\bar{\theta}\bar{u}_{i,i} + \frac{k}{2T_0 p} \bar{\theta}_{,i}\bar{\theta}_{,i} + \frac{p}{2} \bar{m}\bar{\theta}^2 \tag{2.27}$$

or

$$L = -\frac{1}{2} \bar{t}_{ij}\bar{\epsilon}_{ij} - \frac{\rho p^2}{2} \bar{u}_i\bar{u}_i - p\bar{\phi}\bar{\theta}_{,i}\bar{u}_i + \frac{k}{2T_0 p} \bar{\theta}_{,i}\bar{\theta}_{,i} + \frac{p}{2} \bar{m}\bar{\theta}^2 \tag{2.28}$$

provided $\bar{\phi}$ does not depend explicitly on spatial coordinates (x_i). In this case either of (2.27) or (2.28) leads to the field equations (2.9), (2.10), (2.12) and (2.14). The difference between (2.28) and (2.27) is the divergence $(\bar{\theta}\bar{u}_i)_{,i}$. When $\bar{\phi}$ does depend on the x_i, expression (2.27) is the correct Lagrangian to use.

Defining P_{lj} as in (2.23) gives from (2.27) the formula

$$P_{lj} = -\bar{\sigma}_{ij}\bar{u}_{i,l} + \frac{k}{T_0 p} \bar{\theta}_{,j}\bar{\theta}_{,l} - L\delta_{lj} \tag{2.29}$$

or if (2.28) is used P_{lj} is given as

$$P_{lj} = -\bar{t}_{ij}\bar{u}_{i,l} - p\bar{\phi}\bar{u}_j\bar{\theta}_{,l} + \frac{k}{T_0 p} \bar{\theta}_{,j}\bar{\theta}_{,l} - L\delta_{lj}. \tag{2.30}$$

We stress that in (2.29), L is defined by (2.27) whereas in (2.30) L is defined by (2.28).

3. APPLICATIONS OF SECTION 2

3.1 *The coupled thermoviscoelastic strip*

In the following sections, we will consider the plane strain problem of determining the stress intensity factors for a semiinfinite crack in a finite width strip, Fig. 1. From eqn (2.26), the invariant integral

$$F_l = \int_s P_{lj}\,\mathrm{d}s_j \tag{3.1}$$

is zero around any closed path which does not contain a singularity. One such path, ABCDEFG is shown in the figure. For the present purposes, it is sufficient to consider

$$F_1 = \int_s P_{1j}\,\mathrm{d}s_j = 0 \tag{3.2}$$

where P_{lj} is given by eqn (2.29) or (2.30) for an isotropic material. Along the segments BC and GA $\mathbf{n} = \pm\mathbf{e}_2$, and

$$\int_{GA} P_{12}\,\mathrm{d}s_2 = \int_{BC} P_{12}\,\mathrm{d}s_2 \equiv 0 \tag{3.3}$$

since $\bar{u}_i$ and $\bar{\theta}$ are independent of x_1 on the strip sides.

Similarly, along the crack faces, EF and DE, $\mathbf{n} = \pm\mathbf{e}_2$ and from eqns (2.29) and (2.30)

$$P_{12} = -\bar{\sigma}_{i2}\bar{u}_{i,1} \tag{3.4}$$

or

$$P_{12} = -\bar{t}_{i2}\bar{u}_{i,1} \tag{3.5}$$

provided $\bar{\theta}$ does not depend on x_1 on the crack faces. Now for a stress free crack, eqn (3.4) is identically zero. However, eqn (3.5) is seen to be zero only when $\bar{\theta}$ is also zero along the crack face. This restriction must be born in mind when attempting to apply the invariant integral based on eqn (2.30). For this reason, we will use the tensor P_{lj} given by eqn (2.29) in the remainder of this discussion.

The results of eqns (3.3) and (3.4) reduce eqn (3.2) to

$$\oint_{\substack{\text{crack}\\\text{tip}}} P_{1j}\,\mathrm{d}s_j = \int_{AB} P_{1j}\,\mathrm{d}s_j + \int_{CD} P_{1j}\,\mathrm{d}s_j + \int_{FG} P_{1j}\,\mathrm{d}s_j \tag{3.6}$$

where the integral around the crack tip is evaluated on a counter clockwise contour. The result given in eqn (3.6) is particularly useful since the integrals on the r.h.s. may be evaluated for $|x_1| \to \infty$. In this limit, the governing eqns (2.9), (2.10), (2.12) and (2.14) are taken to be independent of x_1 and reduce to a set of coupled ordinary differential equations.

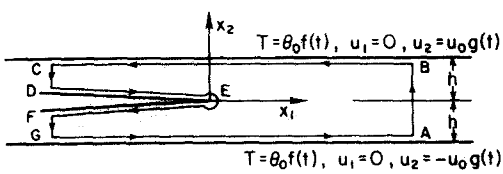

Fig. 1. Integration contour used to determine the stress intensity factor for a semiinfinite crack in a finite width thermoviscoelastic strip.

As $|x_1|\to\infty$, the equilibrium equation (2.9), reduces to

$$\bar{\sigma}_{2i,2}=\rho p^2\bar{u}_i \tag{3.8}$$

with the strain displacement equation given by

$$\bar{\epsilon}_{i2}=\frac{1}{2}(\bar{u}_{i,2}+\bar{u}_{2,i}) \tag{3.9}$$

and the temperature field determined from

$$\frac{k}{T_0}\bar{\theta}_{,22}=p^2\bar{m}\bar{\theta}+p^2\bar{\phi}\bar{\epsilon}_{22}. \tag{3.10}$$

The constitutive eqn (2.12) yields

$$\bar{\sigma}_{i2}=p\bar{G}_1\bar{\epsilon}_{i2}-\frac{p}{3}(\bar{G}_1-\bar{G}_2)\delta_{i2}\bar{\epsilon}_{22}-p\bar{\phi}\delta_{i2}\bar{\theta} \tag{3.11}$$

The first equilibrium equation with (3.11) and (3.9) gives

$$\frac{p}{2}\bar{G}_1\bar{u}_{1,22}=\rho p^2\bar{u}_1 \tag{3.12}$$

with the boundary condition $\bar{u}_1=0$ on $x_2=\pm h$ as $|x_1|\to\infty$. The solution of (3.12) yields $\bar{u}_1=0$. The remaining equilibrium equation yields

$$\left[\frac{p(\bar{G}_2+2\bar{G}_1)}{3}\bar{u}_{2,2}-p\bar{\phi}\bar{\theta}\right]_{,2}=\rho p^2\bar{u}_2 \tag{3.13}$$

where $_{,2}=(\mathrm{d}/\mathrm{d}x_2)$. Equations (3.10) and (3.13) form a system of coupled ordinary differential equations for the temperature field and the displacement at $|x_1|\to\infty$.

The governing equations may be rewritten as

$$[D(p\bar{G}D)-\rho p^2]\bar{u}_2-p\bar{\phi}D\bar{\theta}=0 \tag{3.14}$$

$$(D^2-\bar{\alpha}p^2)\bar{\theta}-p^2\bar{\eta}D\bar{u}_2=0 \tag{3.15}$$

where

$$D=\frac{\mathrm{d}}{\mathrm{d}x_2},\qquad \bar{G}=\frac{\bar{G}_2+2\bar{G}_1}{3},\qquad \bar{\alpha}=\frac{\bar{m}T_0}{k}\quad\text{and}\quad \bar{\eta}=\frac{\bar{\phi}T_0}{k}.$$

The resulting characteristic equation obtained by substituting (3.15) into (3.14) is

$$p\bar{G}D^4-(p^3\bar{\alpha}\bar{G}+\rho p^2+p^3\bar{\phi}\bar{\eta})D^2+p^4\rho\bar{\alpha}=0. \tag{3.16}$$

The roots of (3.16) are found to be

$$\beta_{12}^2=\frac{p(p^2\bar{\alpha}\bar{G}+\rho p+p^2\bar{\phi}\bar{\eta})\pm p\sqrt{((p^2\bar{\alpha}\bar{G}+\rho p+p^2\bar{\phi}\bar{\eta})^2-4p^3\bar{\alpha}\bar{G}\rho)}}{2p\bar{G}} \tag{3.17}$$

and the displacement and temperature fields are

$$\bar{u}_2 = A_1 \sinh \beta_1 x_2 + B_1 \cosh \beta_1 x_2 + A_2 \sinh \beta_2 x_2 + B_2 \cosh \beta_2 x_2 \tag{3.18}$$

$$\bar{\theta} = \frac{\beta_1^2 p\bar{G} - \rho p^2}{\beta_2 p\bar{\phi}}(A_1 \cosh \beta_1 x_2 + B_1 \sinh \beta_1 x_2) + \frac{\beta_2^2 p\bar{G} - \rho p^2}{\beta_2 p\bar{\phi}}(A_2 \cosh \beta_2 x_2 + B_2 \sinh \beta_2 x_2)$$

where A_1, A_2, B_1, and B_2 are determined by the boundary conditions at plus and minus infinity.

Equations (3.18) do not present particular problems in the evaluation of (3.6). However, the subsequent algebra is quite lengthy and will not be presented. In addition, the coupling of the displacement and temperature fields does not allow the stress intensity and thermal intensity (coefficient of singular thermal gradient at crack tip) to be separated in situations where this thermal intensity is non-zero. Thus, in the following section, we will examine the behavior at the crack tip for an uncoupled problem.

3.2 *The uncoupled thermoviscoelastic strip*

Equations (3.14) and (3.15) may be uncoupled by letting $k \to \infty$ while requiring $\bar{\alpha}$ to remain finite. This limit is also obtained for all time if the material is incompressible, i.e. $\epsilon_{kk} = 0$. The characteristic equation (3.16) is then seen to have roots given by

$$\beta_1^2 = p\bar{\alpha}$$

$$\beta_2^2 = \frac{p^2 \rho}{p\bar{G}}. \tag{3.19}$$

We note that β_1 and β_2 are the Laplace transform analogs of the thermal and mechanical wave speeds. Equation (3.18b) reduces to

$$\bar{\theta} = \frac{\beta_1^2 p\bar{G} - \rho p^2}{\beta_1 p\bar{\phi}}(A_1 \cosh \beta_1 x_2 + B_1 \sinh \beta_1 x_2) \tag{3.20}$$

while (3.18a) remains unchanged.

We now consider the boundary conditions as $x_1 \to \infty$. As shown in Fig. 1, the strip is considered to have the same time varying temperature field applied at $x_2 \pm h$ while the edges are displaced by equal and opposite amounts. Thus as $x_1 \to \infty$, we require

$$\bar{\theta} = \theta_0 g(p)$$

$$\bar{u}_2 = \pm u_0 f(p) \quad \text{at} \quad x_2 = \pm h. \tag{3.21}$$

Applying (3.21) to (3.20) and (3.18a) gives the temperature and displacement fields, $x_1 \to \infty$

$$\bar{\theta}(x_2) = \theta_0 g(p)\left(\frac{\cosh \beta_1 x_2}{\cosh \beta_1 h}\right) \tag{3.22}$$

$$\bar{u}_2(x_2) = (u_0 f(p) - C \sinh \beta_1 h)\frac{\sinh \beta_2 x_2}{\sinh \beta_2 h} + C \sinh \beta_1 x_2$$

where

$$C = \frac{p\bar{\phi}\beta_1 \theta_0 g(p)}{p\bar{G}(\beta_1^2 - \beta_2^2)\cosh \beta_1 h}.$$

As $x_1 \to -\infty$, the boundary conditions (3.21) still obtain at $x_2 = \pm h$. However, we must introduce additional conditions along the crack face $x_2 = 0$. Along this line, we will assume the crack is stress free and insulated. These conditions correspond to

$$\bar{\sigma}_{22}(0) = p\bar{G}\bar{u}_{2,2}(0) - p\bar{\phi}\bar{\theta}(0) = 0 \tag{3.23}$$

$$\bar{\theta}_{,2}(0) = 0.$$

Equations (3.20) and (3.18a) then yield as $x_1 \to -\infty$

$$\bar{\theta}(x_2) = \theta_0 g(p) \frac{\cosh \beta_1 x_2}{\cosh \beta_1 h} \tag{3.24}$$

$$\bar{u}_2(x_2) = -\frac{\beta_2}{\beta_1} C \sinh \beta_2 x_2 \pm \left(u_0 f(p) + \frac{\beta_2}{\beta_1} C \sinh \beta_2 h - C \sinh \beta_1 h \right) \frac{\cosh \beta_2 x_2}{\cosh \beta_2 h} + C \sinh \beta_1 h$$

where C is as in (3.22) and $\pm$ applies depending on whether $x_2 > 0$ or $x_2 < 0$.

We note that the temperature fields at $\pm\infty$ are identical. Indeed, for the insulated crack with symmetrically applied temperatures at $\pm h$ the temperature field is independent of x_1 and hence is nonsingular. In addition, the boundary displacements will give rise to only a mode I opening at the crack tip. Thus, we can explicitly determine the stress intensity factor $\bar{K}_I(p)$ from eqn (3.6). Substituting eqns (3.22) and (3.24) into (3.6) gives after some algebra

$$\oint_{\substack{\text{crack}\\ \text{tip}}} \mathrm{P}_{1j}\, \mathrm{d}s_j = \frac{2p\bar{G}\beta_2}{\sinh 2\beta_2 h} \left[u_0 f(p) - \left(\frac{C}{\beta_1}\right)(\beta_1 \sinh \beta_1 h - \beta_2 \sinh \beta_2 h) \right]^2. \tag{3.25}$$

The stresses and displacements near the crack tip are taken to be of the form

$$\bar{\sigma}_{ij} = \frac{\bar{K}_I(p)}{(2\pi r)^{1/2}} f_{ij}(\Theta) \tag{3.26}$$

$$\bar{u}_i = \frac{2\bar{K}_I(p)}{p\bar{G}_1} \left(\frac{2r}{\pi}\right)^{1/2} g_i(\Theta, p\bar{G}, p\bar{G}_2) \tag{3.27}$$

where the functions f_{ij} and g_i have the same angular form as those of the well known elastic case (see Williams[11]). Note that the angle Θ used here is not to be confused with the temperature. The integral around the crack tip can be shown to be

$$\oint_{\substack{\text{crack}\\ \text{tip}}} P_{1j}\, \mathrm{d}s_j = \frac{\bar{K}_I^2(p)}{p\bar{G}_1} \left(\frac{2\bar{G}_1 + \bar{G}_2}{\bar{G}_1 + 2\bar{G}_2} \right). \tag{3.28}$$

Combining (3.25) and (3.28) yields

$$\bar{K}_I(p) = \left[\frac{2}{3} \frac{p\bar{G}_1(p\bar{G}_1 + 2p\bar{G}_2)}{\sinh 2\beta_2 h} \right]^{1/2} \left[u_0 f(p) - \left(\frac{C}{\beta_1}\right)(\beta_1 \sinh \beta_1 h - \beta_2 \sinh \beta_2 h \right] \tag{3.29}$$

for the Laplace transform of the stress intensity factor.

Note that the above result has been derived using the full coupled variational principle and invariant integral (2.27). However, because of the special nature of the boundary condition (3.21), a simple alternative treatment is possible which affords a check on the above result. First the stress distribution in the strip in the absence of the crack is analyzed subject to the specified thermal boundary conditions and in addition the conditions that σ_{22} and u_2 are zero on $x_2 = 0$ for all x_1. This stress and thermal distribution (which depends only on x_2) can thus be subtracted (by superposition) from the original crack problem without violating the boundary conditions on the crack line $x_2 = 0$. This will induce in general a non-zero displacement u_2 on the strip sides. To account for this, one need only solve a viscoelastic (elastic) problem subject to this displacement boundary condition. Such problems of layered media, etc. have been considered by Atkinson[6, 7].

3.3 *Singular temperature gradients*

In the preceeding section, it was possible to calculate the $\bar{K}_I(p)$ explicitly since the temperature gradient at the crack tip was nonsingular. This generally will not be the case. However, when the field equations are uncoupled, it is still possible to compute the magnitudes of the stress singularities by subtracting out the terms associated with the singular temperature gradient. In the following, we consider the problem of determining the thermal intensity factor in a layered composite, Fig. 2. The corresponding elastic problem has been treated by Atkinson[6, 8].

The parts of F_1 in eqns (2.26), (2.27) and (2.29) associated with the thermal field are

$$F_1^{th} = \int_s \left[\frac{k}{T_0 p}\bar{\theta}_{,j}\bar{\theta}_{,1} - \delta_{1j}\left(\frac{k}{2T_0 p}\bar{\theta}_{,i}\bar{\theta}_{,i} + \frac{p\bar{m}}{2}\bar{\theta}^2\right)\right] \mathrm{d}s_j. \tag{3.30}$$

Using arguments similar to those in Section 3.2 and [5, 7] and assuming continuity of flux and temperature at the interface, a result identical to eqn (3.6) is obtained

$$F_1^{th}\Big|_{\substack{\text{crack}\\ \text{tip}}} = F_1|_{x_1\to\infty} + F_1|_{x_1\to-\infty} \tag{3.31}$$

and we need only find the solutions as $|x_1| \to \infty$.

The assumption that the x_1 dependence in the field equations vanishes as $|x_1|\to\infty$ gives a pair of governing equations

$$\bar{\theta}_{,22}^{(i)} = p^2\bar{\alpha}_i\bar{\theta}^{(i)} \quad i = 1, 2 \tag{3.32}$$

for the temperature field. Here $\bar{\alpha}_i = (\bar{m}_i T_0/k_i)$. The solution of eqn (3.32) is

$$\bar{\theta}^{(i)} = A_i \sinh \beta_i x_2 + B_i \cosh \beta_i x_2 \quad i = 1, 2 \tag{3.33}$$

where $\beta_i = \sqrt{(p^2\bar{\alpha}_i)}$. The constants are determined from the boundary conditions

$$\begin{aligned} &\bar{\theta}^{(1)}(\pm h_1 \pm h_2) = \pm\theta_0 f(p) \quad |x_1|\to\infty \\ &\frac{\mathrm{d}\bar{\theta}^{(2)}}{\mathrm{d}n} = \bar{\theta}_{,2}^{(2)}(0) = 0 \quad x_1 \to -\infty \end{aligned} \tag{3.34}$$

and as assumed earlier, the required continuity of flux and temperature at the interface

$$\begin{aligned} &\bar{\theta}^{(1)}(\pm h_2) = \bar{\theta}^{(2)}(\pm h_2) \\ &k_1\bar{\theta}_{,2}^{(1)}(\pm h_2) = k_2\bar{\theta}_{,2}^{(2)}(\pm h). \end{aligned} \tag{3.35}$$

The evaluation of the constants using eqns (3.34) and (3.35) is tedious but straightforward. Having determined the A_i's and B_i's, eqn (3.33) may be used to evaluate the r.h.s. of (3.31). After some algebra, the result is found to be

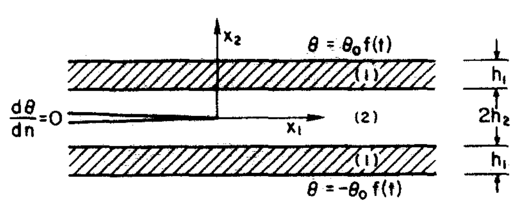

Fig. 2. Finite width, layered thermoviscoelastic strip containing a semi-infinite crack.

$$\Delta^2(F_1|_{x_1\to\infty}+F_1|_{x_1\to-\infty}) = -\frac{\beta_2 k_2}{2T_0 p}\theta_0{}^2 f^2(p)\sinh 2\beta_2 h_2[\cosh^2\beta_1 h_1 - \lambda^2\sinh^2\beta_1 h_1]$$
$$-\frac{\beta_1 k_1}{T_0 p}\theta_0{}^2 f^2(p)\lambda^2\sinh\beta_1 h_1[\cosh\beta_1 h_1\cosh 2\beta_2 h_2$$
$$+\lambda\sinh\beta_1 h_1\sinh 2\beta_2 h_2] \tag{3.36}$$

where

$$\lambda = \frac{k_2\beta_2}{k_1\beta_1}$$

and

$$\Delta = \frac{1}{2}\sinh 2\beta_2 h_2[\cosh^2\beta_1 h_1 + \lambda^2\sinh^2\beta_1 h_1] + \frac{\lambda}{2}\sinh 2\beta_1 h_1\cosh 2\beta_2 h_2.$$

We note that this result coincides with that of Atkinson[7, 8] for a crack in a layered media subjected to harmonic boundary displacements.

The evaluation of F_1 around the crack tip remains. We will take the near tip field to be of the form

$$\bar{\theta} = \bar{K}^{th}(p)\sqrt{\left(\frac{r}{2\pi}\right)}\sin\left(\frac{\Theta}{2}\right). \tag{3.38}$$

This field satisfies the zero flux condition on $\Theta = \pm\pi$ and produces singular fluxes ahead of the crack. Insertion of (3.38) into (3.30) and evaluating as $r\to 0$ gives

$$F_1\Big|_{\substack{\text{crack}\\\text{tip}}} = \frac{-k_2}{T_0 p}\frac{[\bar{K}^{th}(p)]^2}{8}. \tag{3.39}$$

Equations (3.36) and (3.39) give the Laplace transform of the thermal intensity factor for a problem where the crack face is insulated. These results may be subtracted from an uncoupled thermoviscoelastic problem allowing one to obtain the transformed stress intensity factors. Relations similar to (3.36) may also be produced for problems in which the temperature is prescribed along the crack face provided it is not a function of x_1.

4. VARIATIONAL PRINCIPLE AND INVARIANT INTEGRALS FOR A MODEL SYSTEM OF EQUATIONS, AND THEIR APPLICATION TO STEADY STATE THERMOELASTIC PROBLEMS

We consider here a model system which will be seen to apply to steady state thermoelasticity in an appropriate limit. The approach is analogous to that of Section 2 although somewhat simpler. Consider the Lagrangian

$$L = -\frac{1}{2}t_{ij}\epsilon_{ij} + \phi_{ij}\theta u_{i,j} + \frac{k}{2}\theta_{,i}\theta_{,i} \tag{4.1}$$

where

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$$
$$t_{ij} = G_{ijkl}\epsilon_{kl} \quad \text{(anisotropic)} \tag{4.2}$$

and

$$t_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}\text{(isotropic)}.$$

The above Lagrangian depends on u_i, $u_{i,j}$, θ and $\theta_{,i}$ and thus leads to the Euler equations

$$\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial u_{i,j}}\right)=0 \quad \text{and} \quad \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \theta_{,j}}\right)-\frac{\partial L}{\partial \theta}=0 \tag{4.3}$$

(the summation convention with respect to repeated indices is used in the above equations as in earlier sections). In terms of t_{ij}, etc. these Euler equations can be written

$$(-t_{ij}+\phi_{ij}\theta)_{,j}=0 \quad \text{(anisotropic)} \tag{4.4}$$

and

$$k\theta_{,ii}-\phi_{ij}u_{i,j}=0. \tag{4.5}$$

In the isotropic case ϕ_{ij} is replaced by $\phi_0\delta_{ij}$. Following the procedure described earlier we define

$$\begin{aligned} P_{1j} &= \frac{\partial L}{\partial u_{i,j}}u_{i,1}+\frac{\partial L}{\partial \theta_{,j}}\theta_{,1}-L\delta_{1j} \\ &= (-t_{ij}+\phi_{ij}\theta)u_{i,1}+k\theta_{,j}\theta_{,1}-L\delta_{1j} \end{aligned} \tag{4.6}$$

and it can be easily shown that $P_{1j,j}=0$ provided L does not depend explicitly on x_1 and the Euler equations are satisfied.

The field equations (4.4) and (4.5) can be viewed as a "pseudo" set of coupled thermo-elastic equations under time-independent conditions. Putting $k=\infty$ in eqn (4.5) gives the usual steady state equations of thermoelasticity. Moreover, the integral

$$F_1=\int P_{1j}\,\mathrm{d}s_j \tag{4.7}$$

with P_{1j} defined in (4.6) above is path independent since $P_{1j,j}=0$ is consistent with the field equations (4.4) and (4.6) which should tend to the usual (steady) thermoelastic equations when $k\to\infty$. Note, that contributions to F_1 for the integrals taken along a contour such as shown in Fig. 1 will be zero along the crack faces for temperature boundary conditions such as those discussed in Section 3. Note also that it is straightforward to generalize the Lagrangian (4.1) to deal with anisotropic thermal fields such as considered by Atkinson and Clements[12]. The last term in (4.1) is simply replaced by $\frac{1}{2}k_{ij}\theta_{,i}\theta_{,j}$.

4.1 *Applications, cracks in strips. (Isotropic)*

(1) ***Insulated plane crack.*** Suppose the temperature boundary conditions are

$$\begin{aligned} &\theta=\theta_1 \text{ on } x_2=\pm h \text{ for all } x_1 \\ &\frac{\partial \theta}{\partial x_2}=0 \text{ on the crack } x_2=0,\ x_1<0. \end{aligned} \tag{4.8}$$

and in addition displacements are applied to the strip sides, i.e.

$$u_2=\pm u_{20},\ u_1=0 \text{ on } x_2=\pm h \text{ for all } x_1. \tag{4.9}$$

The crack is, of course, stress free so

$$\sigma_{i2}=0 \text{ on } x_2=0,\ x_1<0 \ \ i=1,2. \tag{4.10}$$

The stress being defined as

$$\begin{aligned}\sigma_{ij} &= t_{ij} - \phi_{ij}\theta \text{ (anisotropic)}\\ &= t_{ij} - \phi_0\delta_{ij}\theta \text{ (isotropic).}\end{aligned} \tag{4.11}$$

With the above boundary conditions it is easily seen that P_{12} is zero on the crack faces and on the strip sides $x_2 = \pm h$. We now apply the procedures discussed in Section 3. We consider a contour such as shown in Fig. 1 and evaluate the contributions to F_1 from the path taken through the vertical strips AB and CD at $x_1 \to \infty$ or $-\infty$ respectively. To do this we assume that the boundary conditions are such that variations in field quantities in the x_1 direction are zero at $x_1 = \pm\infty$. With this assumption the field equations (4.4) and (4.5) become (in the isotropic case)

$$\begin{aligned}(-t_{22} + \phi\theta)_{,2} &= 0\\ k\theta_{,22} - \phi u_{2,2} &= 0\end{aligned} \tag{4.12}$$

with

$$t_{22} = (\lambda + 2\mu)u_{2,2}.$$

Solving the coupled eqn (4.12) and substituting into (4.7) gives using the path independence of the integral, after some algebra, the result

$$\oint_{\substack{\text{crack}\\ \text{tip}}} \mathrm{P}_{1j}\,\mathrm{d}s_j = \frac{\phi_0\theta_1 k^{1/2}}{(\lambda+2\mu)^{1/2}}\tanh\left[\frac{\phi_0 h}{k^{1/2}(\lambda+2\mu)^{1/2}}\right] - 2u_{20}\phi_0\theta_1 + \frac{\phi_0 u_{20}^2(\lambda+2\mu)^{1/2}}{k^{1/2}}\coth\left[\frac{\phi_0 h}{k^{1/2}(\lambda+2\mu)^{1/2}}\right]. \tag{4.14}$$

The l.h.s. of this equation can be easily evaluated around a vanishing small circle at the crack tip to give

$$\oint_{\substack{\text{crack}\\ \text{tip}}} P_{1j}\,\mathrm{d}s_j = K_1{}^2\frac{(1-\nu)}{2\mu} \tag{4.15}$$

where from symmetry the only singular terms at the crack tip are due to the usual $r^{-1/2}$ singularities in the crack tip stresses and

$$\sigma_{22} = (2\pi)^{-1/2}K_1 r^{-1/2} \text{ as } r \to 0 \text{ on } x_2 = 0, x_1 > 0. \tag{4.16}$$

Letting $k \to \infty$ in eqn (4.14) gives using (4.15)

$$\lim_{k\to\infty} K_1^2\frac{(1-\nu)}{2\mu} = \frac{(\lambda+2\mu)}{h}\left[\frac{\phi_0\theta_1 h}{(\lambda+2\mu)} - u_{20}\right]^2$$

and hence

$$K_1\left(\frac{1-\nu}{2\mu}\right)^{1/2} = \frac{(\lambda+2\mu)^{1/2}}{h^{1/2}}\left(u_{20} - \frac{\phi_0\theta_1 h}{(\lambda+2\mu)}\right) \tag{4.18}$$

which ought to be the solution for the corresponding steady thermoelastic problem (i.e. with the temperature equation $\theta_{,jj} = 0$). The result (4.18) agrees with the result obtained in Section 3 as the steady state limit $p \to 0$ of the solution given there.

Note, that since we are interested in the steady thermoelastic situation one could proceed by solving eqns (4.12) at $x_1 \to \pm\infty$ in the limit $k \to \infty$. Doing this for the situation where ϕ_0 and

$(\lambda + 2\mu)$ may vary with x_2 (symmetrically about $x_2 = 0$ for convenience) leads to the result

$$\lim_{k\to\infty} K_1\left(\frac{1-\nu}{2\mu}\right)_0^{1/2} = \left[u_{20} - \theta_1 \int_0^h \frac{\phi_0\,\mathrm{d}x_2}{(\lambda+2\mu)}\right] \Big/ \left[\int_0^h \frac{\mathrm{d}x_2}{(\lambda+2\mu)}\right]^{1/2} \tag{4.19}$$

and the subscript 0 on $(1-\nu)/2\mu$ is to indicate that these moduli are evaluated on $x_2 = 0$. It should be emphasized that the full (isotropic) Lagrangian of eqn (4.1) has been used in these calculations, only the far field $(x_1 \to \pm\infty)$ calculations have been simplified with the assumption $k \to \infty$ to deduce eqns (4.19).

(2) *Insulated crack disturbing a uniform temperature gradient.* For this case the temperature boundary conditions are

$$\begin{aligned} \theta &= \pm\,\theta_1 \quad \text{on} \quad x_2 = \pm\, h \\ \frac{\partial\theta}{\partial x_2} &= 0 \text{ on the crack } x_2 = 0,\ x_1 < 0. \end{aligned} \tag{4.20}$$

Since the temperature field is an odd function of x_2 the only non-zero stress induced on the plane $x_2 = 0$ in the absence of the crack will be a shear stress.

The boundary conditions are

$$u_2 = 0 = u_1 \text{ on } x_2 = \pm\, h \text{ for all } x_1. \tag{4.21}$$

and since the crack is stress free

$$\sigma_{i2} = 0 \text{ on } x_2 = 0,\ x_1 < 0. \tag{4.22}$$

For the coupled case a similar calculation to that shown in Example 1 gives

$$\begin{aligned} \oint_{\substack{\text{crack}\\\text{tip}}} P_{1j}\,\mathrm{d}s_j &= -\frac{2\phi_0(k(\lambda+2\mu))^{1/2}\theta_1^{\,2}}{(\lambda+2\mu)\sinh\left(\dfrac{2\phi_0 h}{(k(\lambda+2\mu))^{1/2}}\right)} \\ &= -\frac{\theta_1^2 k}{h} + \frac{2}{3}\frac{\phi_0^2 h\theta_1^2}{(\lambda+2\mu)} + 0(1/k) \text{ as } k\to\infty. \end{aligned} \tag{4.23}$$

Note that as $k\to\infty$ in eqn (4.5) this equation reduces to the usual steady state heat equation, i.e. $\theta_{,jj} = 0$. The solution of this equation subject to the boundary conditions (4.20) can be found by the methods of Section 3 (compare Section 3.3 as p tends to zero). Thus the thermal intensity factor can be found separately in this limit, its use in evaluating the l.h.s. of (4.23) leads to the term $-\theta_1^{\,2}k/h$. The stress-intensity factor in the limit $k\to\infty$ can then be deduced from the second term in $(4.23)_2$ and leads to the result

$$K_{II} = \phi_0\theta_1\left(\frac{4\mu h}{(1-\nu)(\lambda+2\mu)3}\right)^{1/2} \tag{4.24}$$

for the shear stress intensity factor induced by the thermal field in the steady thermoelastic limit.

5. CONCLUDING REMARKS

Invariant integrals have been derived (Section 2) and are valid for time-dependent coupled thermoviscoelasticity even when the viscoelastic moduli, thermal diffusivity etc. vary spatially in directions perpendicular to the crack plane. In Section 3 we have applied these integrals to certain crack problems in a thermoviscoelastic strip. For certain boundary conditions it is possible to determine explicit crack tip stress and displacement fields using such invariant integrals and calculations made as x_1 tends to plus and minus infinity in the strip (see Fig. 1). Although the procedure outlined in Section 3 is inherently simple, in practice it can be algebraically complicated and moreover gives explicit results in terms of the Laplace transform, so more work is required to determine real time results.

It is relatively easy, however, to extract results for small and large times from the full Laplace transformed results by means of various Tauberian theorems. For example, the large time behavior is related to the behavior of the transform as the transform parameter p tends to zero. Thus in the results of Section 3.2, e.g. if we let p tend to zero we obtain results in agreement with those of Section 4.1, Example 1, if $f(p)$ and $g(p)$ are chosen so that they tend to infinity like $1/p$ as p tends to zero. Such a choice is consistent with the existence of the steady state.

It should be noted that the procedure we have outlined can also be applied to other field equations.

For example, the results of Section 2 may also be used to derive invariant integrals for the non-simple elastic materials[13]. The governing equations for the infinitesimal theory are[13]

$$\frac{k}{T_0} T_{,jj} - \phi \dot{u}_{i,i} - m(\dot{T} - a\dot{T}_{,kk}) = 0 \tag{5.1}$$

$$\mu u_{i,kk} + (\lambda + \mu) u_{j,ji} - \phi(T_{,i} - aT_{,kki}) = \rho \ddot{u}_i. \tag{5.2}$$

In (5.1) and (5.2), λ and μ are the Lame moduli, a is the temperature discrepancy, and T is the deviation of the conductive temperature from a constant reference T_0. Also in (5.1) and (5.2), we have taken the heat supply and body force to be zero. Laplace transforming eqns (5.1) and (5.2) gives

$$\frac{k}{T_0} \bar{T}_{,jj} - p\phi \bar{u}_{i,i} - m(p\bar{T} - ap\bar{T}_{,kk}) = 0 \tag{5.3}$$

$$\mu \bar{u}_{i,kk} + (\lambda + \mu) \bar{u}_{j,ji} - \phi(\bar{T}_{,i} - a\bar{T}_{,kki}) = \rho p^2 \bar{u}_i. \tag{5.4}$$

For $a = 0$, eqns (5.3) and (5.4) are simply the equations of coupled thermoelasticity and the Lagrangian as given by (2.27) is

$$L = -\frac{1}{2} \bar{t}_{ij}\bar{\varepsilon}_{ij} - \frac{\rho p^2}{2} \bar{u}_i \bar{u}_i + \phi \bar{T} \bar{u}_{i,i} + \frac{k}{2T_0 p} \bar{T}_{,i}\bar{T}_{,i} + \frac{m}{2} \bar{T}^2 \tag{5.5}$$

where

$$\bar{t}_{ij} = \lambda \delta_{ij} \bar{\varepsilon}_{kk} + 2\mu \bar{\varepsilon}_{ij}.$$

The corresponding "energy—momentum tensor" is found in eqn (2.29)

$$P_{lj} = -\bar{\sigma}_{ij}\bar{u}_{i,l} + \frac{k}{T_0} \bar{T}_{,j}\bar{T}_{,l} - L\delta_{lj} \tag{5.6}$$

where

$$\bar{\sigma}_{ij} = \bar{t}_{ij} - \phi \delta_{ij} \bar{T}.$$

We note that eqns (2.28) and (2.30) may also be used to define an invariant integral. However, it should be recognized that P_{lj} in eqn (2.30) may not be zero along the crack faces. The condition of zero stress along the crack faces requires $\bar{\sigma}_{i2}$ be zero but not $\bar{t}_{i2}$.

For the full eqns (5.3) and (5.4), a Lagrangian may be sought by adding a term like $\phi a \bar{T}_{,ij} \bar{u}_{i,j}$ to (5.5). Such a term will give the correct field equation for displacement (5.4), but the Euler equation for the temperature

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{\partial L}{\partial \bar{T}_{,ij}}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial \bar{T}_{,i}}\right) + \frac{\partial L}{\partial \bar{T}} = 0 \tag{5.7}$$

does not yield equation (5.3). However, equations (5.3) may be rearranged to give

$$\left(\frac{k}{T_0} + pma\right)\bar{T}_{,jj} - p\phi\bar{u}_{i,i} - mp\bar{T} = 0. \tag{5.8}$$

Substituting for $\bar{T}_{,kki}$ in eqn (5.4) gives

$$\mu\bar{u}_{i,kk} + \left(\lambda + \frac{\phi^2}{\dfrac{k}{paT_0} + m} + \mu\right)\bar{u}_{j,ji} - \phi\left(1 - \frac{1}{\dfrac{k}{pmaT_0} + 1}\right)\bar{T}_{,i} = \rho p^2\bar{u}_i. \tag{5.9}$$

It is recognized that the displacement terms may be derived from a pseudo-elastic strain energy density function by defining a constant

$$\lambda' = \lambda + \frac{\phi^2}{\dfrac{k}{paT_0} + m}.$$

Thus eqn (5.9) can be written as

$$\mu\bar{u}_{i,kk} + (\lambda' + \mu)\bar{u}_{j,ji} - \eta\bar{T}_{,i} = \rho p^2\bar{u}_i \tag{5.10}$$

where

$$\eta = \phi\left(1 - \frac{1}{\dfrac{k}{pmaT_0} + 1}\right).$$

We see that eqns (5.8) and (5.9) are of the same form as the coupled equations for simple thermoelastic materials and the Lagrangian is

$$L = -\frac{1}{2}\bar{t}'_{ij}\bar{\varepsilon}_{ij} - \frac{\rho p^2}{2}\bar{u}_i\bar{u}_i + \eta\bar{T}\bar{u}_{i,i} + \frac{\eta}{\phi}\left(\frac{k'}{2T_0p}\bar{T}_{,i}\bar{T}_{,i} + \frac{m}{2}\bar{T}^2\right)$$

where

$$\bar{t}'_{ij} = \lambda'\delta_{ij}\bar{\varepsilon}_{kk} + 2\mu\bar{\varepsilon}_{ij} \tag{5.11}$$

$$k' = k + pmaT_0.$$

The "energy-momentum tensor" for eqn (5.11) is

$$P_{lj} = -\bar{\sigma}'_{ij}\,\bar{u}_{i,l} + \frac{k'\eta}{pT_0\phi}\bar{T}_{,j}\bar{T}_{,l} - L\delta_{lj} \tag{5.12}$$

where

$$\bar{\sigma}'_{ij} = \bar{t}'_{ij} - \eta\delta_{ij}\bar{T}.$$

Thus the results in Section 3 apply equally well to the non-simple thermoelastic materials with appropriate definition of the material properties.

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